A brief introduction to the discrete Morse flow

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0.1 The variational method

The approximation method that we will introduce goes by several different names, but it is a minimizing movement that is intimately related to the method of Rothe. It is often referred to as *the discrete Morse flow* (DMF), or as a *minimizing movement* (MM).

We will thus describe the method of Rothe as applied to the heat equation and then discuss discrete gradient descents. Finally we will introduce the DMF.

0.2 The method of Rothe

Suppose we want to find a solution to the heat equation:

$$\begin{cases} u_t = \Delta u + f(t,x) & \text{in } (0,T)\times\Omega, \\ u|_{\partial\Omega} = g(t,x) & \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{1}$$

where Ω is a domain in $\mathbf{R}^d$, $T > 0$ is a final time, and $f(t,x) \in C(\bar{\Omega}_T)$ is a given source term and g is the boundary data. The method of Rothe takes the position of approximating solutions to (1) by discretizing time and then looking at the solution to time-local approximations of (1).

For the sake of simplicity, let us assume that $g(t,x) = 0$ and let u_0 be a given function from $H_0^1(\Omega))$. Further, let $h = T/M$ denote a discretized time step, for some integer $M > 1$. Then, leaving the space variable continuous, consider approximating the time-local evolution by the solution to the following elliptic problem:

$$\begin{cases} \frac{u-u_0}{h} = \Delta u + f(h,x) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

By standard elliptic theory there exists a unique solution to the above, which we denote by u_1. Then, in the same way as for u_0, we can thus build a sequence of functions $\{u_n\}_{n=0}^M$ such that each u_n ($n \geq 1$) is the solution of a corresponding elliptic problem:

$$\begin{cases} \frac{u-u_{n-1}}{h} = \Delta u + f(nh,x) & \text{in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{3}$$

In particular, for any $\phi \in H_0^1(\Omega)$, each member of the sequence of functions satisfies

$$\int_\Omega \frac{u_n - u_{n-1}}{h}\phi + \nabla u_n \nabla\phi dx = \int_\Omega f(nh,x)\phi dx. \tag{4}$$

Solutions to the elliptic problems can then be interpolated in time to obtain an approximate solution to the original problem on all of Ω_T.

For each $k = 1, 2, ..., M$ one defines χ_k to be the characteristic function of the time interval $[(k-1)h, kh)$:

$$\chi_k(t) = \begin{cases} 1 & \text{if } t \in [(k-1)h, kh), \\ 0 & \text{otherwise.} \end{cases}$$

A piecewise linear time interpolation with parameter h is then defined by

$$u^h(t,x) = \sum_{k=1}^{M} \chi_k(t) \left[\left(\frac{t-(k-1)h}{h} \right) u_k(x) + \left(\frac{kh-t}{h} \right) u_{k-1}(x) \right],$$

and a piecewise constant step function by (see figure 1):

$$\bar{u}^h(t,x) = \sum_{k=1}^{M} \chi_k(t) u_k(x).$$

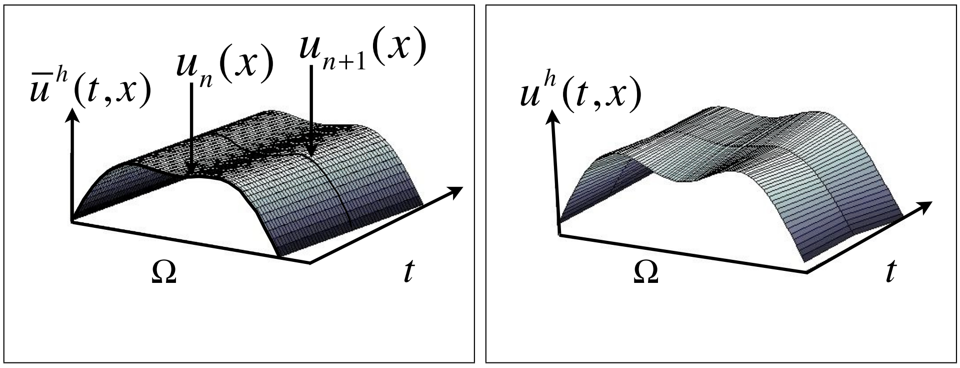

Figure 1: Piecewise constant $\bar{u}^h$ (left) and piecewise linear u^h (right).

For the discretized source term considered here, we write

$$\bar{f}^h(t,x) = \sum_{k=1}^{M} \chi_k(t) f(kh,x). \tag{5}$$

Then for any $t \in (0,T)$ and $\phi \in H_0^1(\Omega)$, one has

$$\int_\Omega \left(u_t^h \phi + \nabla \bar{u}^h \nabla \phi \right) dx = \int_\Omega \bar{f}^h(t,x) \phi dx. \tag{6}$$

By taking $\phi(x) = u_n - u_{n-1}$ in (4) and summing from $n = 0$ to M, one readily obtains the following estimates:

$$\begin{aligned} \left|\left|u_t^h\right|\right|^2_{L^2(\Omega_T)} &\leq ||\nabla u_0||^2_{L^2(\Omega)} + \left|\left|\bar{f}^h\right|\right|^2_{L^2(\Omega_T)} \\ \left|\left|\nabla u^h\right|\right|^2_{L^2(\Omega_T)} &\leq C(||\nabla u_0||^2_{L^2(\Omega)} + \left|\left|\bar{f}^h\right|\right|^2_{L^2(\Omega_T)}) \\ \left|\left|\bar{u}^h - u^h\right|\right|^2_{L^2(\Omega_T} &\leq h^2(||\nabla u_0||^2_{L^2(\Omega)} + \left|\left|\bar{f}^h\right|\right|^2_{L^2(\Omega_T)}), \end{aligned}$$

where C is a constant that is independent of h. The assumption that the source function belongs to $C(\bar{\Omega}_T)$ implies that $\bar{f}^h$ converges to f in $L^2(\Omega_T)$. Therefore,

$$\lim_{h \downarrow 0} \left|\left|\bar{f}^h\right|\right|^2_{L^2(\Omega_T)} = ||f||^2_{L^2(\Omega_T)}. \tag{7}$$

In particular, one has that $\{u^h\}$ is bounded in $W^{1,2}(\Omega_T)$, so that there exists a function $u \in W^{1,2}(\Omega_T)$ and a subsequence $\{u^{h_j}\}$ such that u^{h_j} converges to u, and $u_t^{h_j}$ and ∇u^{h_j} converge weakly to u_t and ∇u, respectively, in $L^2(\Omega_T)$. Moreover, one has $u \in W_0^{1,2}([0,T)\times\Omega)$ and that the trace of $u(0,x)$ equals u_0 almost everywhere.

One extends the test functions to time-dependent domains and integrates (6) in time. Then for any $\phi \in C_0^\infty([0,T)\times\Omega)$, the following holds:

$$\int_0^T \int_\Omega \left(u_t^h \phi + \nabla \bar{u}^h \nabla \phi\right) dxdt = \int_0^T \int_\Omega \bar{f}^h \phi dxdt. \tag{8}$$

Using the convergence information and the fact that $u \in W_0^{1,2}([0,T)\times\Omega)$, one is thus able to obtain a weak solution to the original problem, as h goes to zero:

$$\int_0^T \int_\Omega (u_t \phi + \nabla u \nabla \phi)\, dxdt = \int_0^T \int_\Omega f \phi dxdt. \qquad (\forall \phi(t,x) \in C_0^\infty([0,T)\times\Omega)).$$

0.2.1 Time-discrete gradient descents

The method just described can also be interpreted as a time-discretized gradient descent of an energy functional. We will now describe this relationship and introduce the discrete Morse flow.

Let $\mathcal{E}(u)$ denote the Dirichlet integral of a function $u \in H_0^1(\Omega)$:

$$\mathcal{E}(u) = \int_\Omega \frac{|\nabla u|^2}{2} dx. \tag{9}$$

Parallel to the finite dimensional case, the gradient of this energy at a location u in $L^2(\Omega)$ is defined as the function G such that,

$$\frac{d}{dt}E(u) = (G, u_t)_{L^2}. \tag{10}$$

We compute:

$$\begin{aligned} \frac{d}{dt}E(u) =& \frac{1}{2}\int_\Omega \frac{d}{dt}\nabla u \cdot \nabla u dx \\ =& \int_\Omega \nabla u_t \cdot \nabla u dx \\ =& \int_{\partial\Omega} u_t \nabla u \cdot \nu dS - \int_\Omega u_t \Delta u dx \\ =& (-\Delta u, u_t)_{L^2}, \end{aligned} \tag{11}$$

where we have assumed that u is subject to Dirichlet boundary conditions. Thus we see that the gradient of the Dirichlet integral at u in L^2 is the minus Laplacian:

$$\nabla \mathcal{E}(u) = -\Delta u.$$

Now let u_0 be a given function, and consider a time-discrete approximation of the gradient descent (in L^2) of $\mathcal{E}(u)$:

$$\begin{aligned} u =& u_0 - h\nabla\mathcal{E}(u_0) \\ =& u_0 + h\Delta u_0. \end{aligned}$$

Which is equivalent to

$$\frac{u - u_0}{h} = \Delta u_0. \tag{12}$$

By formally decreasing h to zero we have:

$$u_t = \Delta u,$$

which is often interpreted as saying that "the gradient descent of the Dirichlet integral is the heat equation."

Comparing the descent (12) to that of (2), we see that the method of Rothe can be interpreted as a time-discrete gradient descent of the Dirichlet integral. With this in mind, we now discuss the idea behind the DMF.

0.2.2 The discrete Morse flow

Parallel to the method of Rothe, let $h = T/M$ denote a time discretization. For each $n = 1, ..., M$, we consider the minimization of a functional $\mathcal{F}_n(u)$, defined over $H_0^1(\Omega)$:

$$\mathcal{F}_n(u) = \int_\Omega \left(\frac{|u - u_{n-1}|^2}{2h} + \frac{|\nabla u|^2}{2} - f(hn, x)u \right) dx. \tag{13}$$

By computing the first variation of $\mathcal{F}_n(u)$:

$$\frac{d}{d\epsilon}\mathcal{F}_n(u + \epsilon\phi)|_{\epsilon=0} = 0, \tag{14}$$

we deduce an expression for a weak solution to the elliptic problem (3):

$$\int_\Omega \frac{u - u_{n-1}}{h}\phi + \nabla u \nabla \phi dx = \int_\Omega f(nh, x)\phi dx.$$

The existence of unique minimizers for each $\mathcal{F}_n(u)$ can be shown by the direct method. The weak lower semicontinuity of the functionals, together with the boundedness of minimizing sequences constitute the significant points of the proof.

We then proceed as in the method of Rothe. By constructing a sequence of functions $\{u_n\}_{n=0}^\infty$, each the unique minimizer of a functional $\mathcal{F}_n(u)$ $(n \geq 1)$, we are able to approximate the evolution by (1) at discrete times $t = 0, h, 2h, ..., Mh$. Interpolating the minimizers in time and estimating the approximate solution allows one to take the interpolation parameter to zero to obtain a solution to the original problem.

Of course, once one has shown the existence and uniqueness of minimizers, the solutions obtained both methods are equivalent. Nevertheless, the variational nature of the DMF has a few distinct advantages.

The first is that the minimization aspect imparts the DMF's numerical method with strong stability properties, and another is that it allows one to easily consider constrained evolutions (by penalization, or by explicitly restricting the admissible function set for the minimizations). Finally, computations for problems involving free boundaries can be addressed in a relatively straightforward manner. In particular, the minimizations automatically determine the location of the free boundaries as well as the solution to the target problem.

0.3 Numerical Implementation of the DMF

We will give the details for using the DMF as a numerical method. The calculations corresponding to the DMF will assume the case $d = 3$ (that is, $\Omega \subset \mathbf{R}^3$, as the lower dimensional derivations can also be understood from this approach.

0.4 Volume coordinates

In this section, we derive formulae for computing functional values under the $\mathbf{P}^1$ finite element assumptions. We assume that a domain $\Omega \subset \mathbf{R}^d$ has been partitioned into a finite number of elements. This can be done in any number of ways, but Delaunay triangulations are the most common. This method takes a finite collection of points from within the domain and then creates a corresponding triangulation (a graph). See figure 2 for an illustration of this process.

Let us now describe the finite element approximation of our functions.

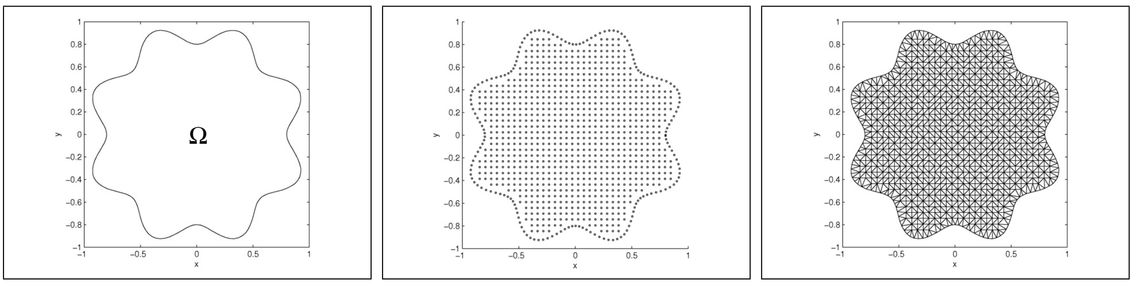

Figure 2: (Left) The domain Ω. (Center) Points in the domain. (Right) The triangulation.

To this end, let $K-1$ denote the dimension of vector $\boldsymbol{u}$ and e be the tetrahedron with vertices located at $\boldsymbol{x}_i = (x_i, y_i, z_i), i = 1, ..., 4$ (see figure 3).

The $\mathbf{P}^1$ assumptions imply that each coordinate of the vector field $\boldsymbol{u}$ can be written

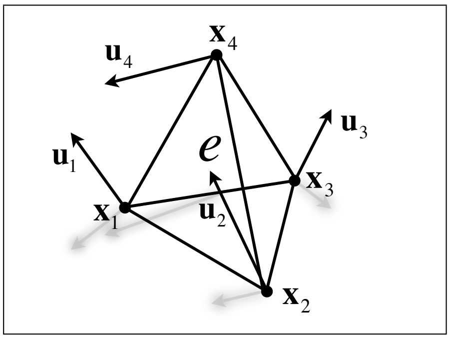

Figure 3: A tetrahedral element with it vertices and vectors.

as follows, over an arbitrary tetrahedron:

$$\begin{pmatrix} u_1^k \\ u_2^k \\ u_3^k \\ u_4^k \end{pmatrix} = \begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix} \begin{pmatrix} \alpha^k \\ \beta^k \\ \gamma^k \\ \zeta^k \end{pmatrix} \qquad (k = 1, ..., K-1), \tag{15}$$

where u_i^k denotes the k^{th} element of the vector field $\boldsymbol{u}$ at location (x_i, y_i, z_i). Further, let D denote the following determinant:

$$D = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}. \tag{16}$$

Cramer's rule allows one to express the coefficients in (15) in the following way:

$$D\alpha^k = \begin{vmatrix} u_1^k & y_1 & z_1 & 1 \\ u_2^k & y_2 & z_2 & 1 \\ u_3^k & y_3 & z_3 & 1 \\ u_4^k & y_4 & z_4 & 1 \end{vmatrix} \qquad D\beta^k = \begin{vmatrix} x_1 & u_1^k & z_1 & 1 \\ x_2 & u_2^k & z_2 & 1 \\ x_3 & u_3^k & z_3 & 1 \\ x_4 & u_4^k & z_4 & 1 \end{vmatrix} \tag{17}$$

$$D\gamma^k = \begin{vmatrix} x_1 & y_1 & u_1^k & 1 \\ x_2 & y_2 & u_2^k & 1 \\ x_3 & y_3 & u_3^k & 1 \\ x_4 & y_4 & u_4^k & 1 \end{vmatrix} \qquad D\zeta^k = \begin{vmatrix} x_1 & y_1 & z_1 & u_1^k \\ x_2 & y_2 & z_2 & u_2^k \\ x_3 & y_3 & z_3 & u_3^k \\ x_4 & y_4 & z_4 & u_4^k \end{vmatrix}. \tag{18}$$

We now change to the so-called *volume coordinates* by inverting the matrix on the right hand side of (15) to write:

$$\begin{pmatrix} \alpha_1^k \\ \beta_1^k \\ \gamma_1^k \\ \zeta_1^k \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix} \begin{pmatrix} u_1^k \\ u_2^k \\ u_3^k \\ u_4^k \end{pmatrix}. \tag{19}$$

We compute α^k:

$$
\begin{aligned}
D\alpha^k &= u_1^k \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - u_2^k \begin{vmatrix} y_1 & z_1 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} + u_3^k \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - u_4^k \begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \\
&= -(u_4^k \left(y_1(z_2 - z_3) - y_2(z_1 - z_3) + y_3(z_1 - z_2)\right) \\
&\quad -u_3^k \left(y_1(z_2 - z_4) - y_2(z_1 - z_4) + y_4(z_1 - z_2)\right) \\
&\quad +u_2^k \left(y_1(z_3 - z_4) - y_3(z_1 - z_4) + y_4(z_1 - z_3)\right) \\
&\quad -u_1^k \left(y_2(z_3 - z_4) - y_3(z_2 - z_4) + y_4(z_2 - z_3)\right)) \\
&=: -(u_4^k a_1 - u_3^k a_2 + u_2^k a_3 - u_1^k a_4). \qquad (20)
\end{aligned}
$$

Similarly, for β^k, γ^k, and ζ^k, we have

$$
\begin{aligned}
D\beta^k &= -(u_4^k \left(x_1(z_2 - z_3) - x_2(z_1 - z_3) + x_3(z_1 - z_2)\right) \\
&\quad -u_3^k \left(x_1(z_2 - z_4) - x_2(z_1 - z_4) + x_4(z_1 - z_2)\right) \\
&\quad +u_2^k \left(x_1(z_3 - z_4) - x_3(z_1 - z_4) + x_4(z_1 - z_3)\right) \\
&\quad -u_1^k \left(x_2(z_3 - z_4) - x_3(z_2 - z_4) + x_4(z_2 - z_3)\right)) \\
&=: -(u_4^k b_1 - u_3^k b_2 + u_2^k b_3 - u_1^k b_4), \qquad (21)
\end{aligned}
$$

$$
\begin{aligned}
D\gamma^k &= -(u_4^k \left(x_1(y_2 - y_3) - x_2(y_1 - y_3) + x_3(y_1 - y_2)\right) \\
&\quad -u_3^k \left(x_1(y_2 - y_4) - x_2(y_1 - y_4) + x_4(y_1 - y_2)\right) \\
&\quad +u_2^k \left(x_1(y_3 - y_4) - x_3(y_1 - y_4) + x_4(y_1 - y_3)\right) \\
&\quad -u_1^k \left(x_2(y_3 - y_4) - x_3(y_2 - y_4) + x_4(y_2 - y_3)\right)) \\
&=: -(u_4^k c_1 - u_3^k c_2 + u_2^k c_3 - u_1^k c_4), \qquad (22)
\end{aligned}
$$

$$
\begin{aligned}
D\zeta^k &= -(u_4^k \left(x_1(y_2 z_3 - y_3 z_2) - x_2(y_1 z_3 - y_3 z_1) + x_3(y_1 z_2 - y_2 z_1)\right) \\
&\quad -u_3^k \left(x_1(y_2 z_4 - y_4 z_2) - x_2(y_1 z_4 - y_4 z_1) + x_4(y_1 z_2 - y_2 z_1)\right) \\
&\quad +u_2^k \left(x_1(y_3 z_4 - y_4 z_3) - x_3(y_1 z_4 - y_4 z_1) + x_4(y_1 z_3 - y_3 z_1)\right) \\
&\quad -u_1^k \left(x_2(y_3 z_4 - y_4 z_3) - x_3(y_2 z_4 - y_4 z_2) + x_4(y_2 z_3 - y_3 z_2)\right)) \\
&=: -(u_4^k d_1 - u_3^k d_2 + u_2^k d_3 - u_1^k d_4). \qquad (23)
\end{aligned}
$$

Over each tetrahedron, the vector field is written:

$$\boldsymbol{u}(x, y, z) = \bar{\alpha} x + \bar{\beta} y + \bar{\gamma} z + \bar{\zeta},$$

where $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ and $\bar{\zeta}$ denote the vectors with coordinates $\alpha^k, \beta^k, \gamma^k$ and ζ^k, respectively. Hence,

$$u^k(x, y, z) = \sum_{i=1}^{4} \left(a_i x + b_i y + c_i z + d_i\right) u_i^k.$$

Set $\lambda_i = a_i x + b_i y + c_i z + d_i$, so that the above is expressed:

$$u^k(x,y,z) = \sum_{i=1}^{4} \lambda_i u_i^k. \tag{24}$$

Then

$$\frac{\partial u^k}{\partial x} = \sum_{i=1}^{4} a_i u_i^k, \quad \frac{\partial u^k}{\partial y} = \sum_{i=1}^{4} b_i u_i^k, \quad \frac{\partial u^k}{\partial z} = \sum_{i=1}^{4} c_i u_i^k. \tag{25}$$

Now we can use the value of the well-known volume coordinates integral to compute the value of our functionals:

$$\int_e \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma \lambda_4^\delta dxdydz = \frac{\alpha!\beta!\gamma!\delta!}{(\alpha+\beta+\gamma+\delta+3)!} 6\,|e|\,. \tag{26}$$

Remark: In the two-dimensional case, the equivalent of the above is as follows:

$$\int_e \lambda_1^\alpha \lambda_2^\beta \lambda_3^\gamma dxdy = \frac{\alpha!\beta!\gamma!}{(\alpha+\beta+\gamma+2)!} 2\,|e|\,.$$

Let w_k denote the value of a $\mathbf{P}^1$ function at the location k. Then we have

$$\begin{aligned}(w_1\lambda_1 + w_2\lambda_2 + w_3\lambda_3 + w_4\lambda_4)^2 =& w_1^2\lambda_1^2 + w_2^2\lambda_2 + w_3^2\lambda_3^2 + w_4^2\lambda_4^2 \\ &+2w_1w_2\lambda_1\lambda_2 + w_1w_3\lambda_1\lambda_3 + w_1w_4\lambda_1\lambda_4 \\ &+2w_2w_3\lambda_1\lambda_3 + w_1w_3\lambda_1\lambda_3 + w_3w_4\lambda_3\lambda_4 \\ &+2w_2w_4\lambda_2\lambda_4 + w_1w_4\lambda_1\lambda_4 + w_3w_4\lambda_3\lambda_4. \end{aligned} \tag{27}$$

By using (26) we compute the following values:

$$\int_e \lambda_i \lambda_j dxdydz = \begin{cases} |e|\,/10 & \text{if } i = j \\ |e|\,/20 & \text{otherwise.} \end{cases}$$

Therefore the values of our functionals over a each element can be computed using the following formula:

$$\int_e (w_1\lambda_1 + w_2\lambda_2 + w_3\lambda_3 + w_4\lambda_4)^2 dxdydz \tag{28}$$

$$= \frac{|e|}{10} \left(w_1^2 + w_2^2 + w_3^2 + w_4^2 + w_1w_2 + w_2w_3 + w_2w_4 + w_1w_3 + w_1w_4 + w_3w_4 \right). \tag{29}$$

For the Dirichlet integral, we have:

$$\begin{aligned}\int_e |\nabla \boldsymbol{w}|^2 \, dxdydz &= \int_e \sum_{i=1}^{K-1} |\nabla w^i|^2 \, dxdydz \\ &= \int_e \sum_{i=1}^{K-1} \left| \nabla \left(\sum_{j=1}^{4} \lambda_j w_j^i \right) \right|^2 dxdydz \\ &= |e| \sum_{i=1}^{K-1} \left[\left(\sum_{j=1}^{4} w_j^i a_j \right)^2 + \left(\sum_{j=1}^{4} w_j^i b_j \right)^2 + \left(\sum_{j=1}^{4} w_j^i c_j \right)^2 \right]. \end{aligned} \tag{30}$$

Remark: It is important to realize that, for an arbitrary vector field, the above formulae allow one to compute functional values in terms of the mesh geometry. In particular, one needs compute the area and coefficients for each element only once. Then one can reuse their information for any other candidate vector field, throughout the computations.

0.5 Computation of functional values

For the sake of clarity, we explicitly state the way of computing functional values for vector-type discrete Morse flows, with $\Omega \subset \mathbf{R}^2$.

The task is to find a vector field $\boldsymbol{u} \in H_0^1(\Omega; \mathbf{R}^{K-1})$ to minimize functionals of the following type:

$$\mathcal{F}_n(\mathbf{u}) = \int_\Omega \left(\frac{|\boldsymbol{u} - 2\boldsymbol{u}_{n-1} + \boldsymbol{u}_{n-2}|^2}{2h^2} + \frac{|\boldsymbol{u} - \boldsymbol{u}_{n-1}|^2}{2h} + \frac{|\nabla \boldsymbol{u}|^2}{2} \right) dx,$$

where $h > 0$ and $\boldsymbol{u}_{n-1}, \boldsymbol{u}_{n-2}$ are given.

Assuming the $\mathbf{P}^1$ finite element assumptions, over each element $\boldsymbol{u}$ has the following form:

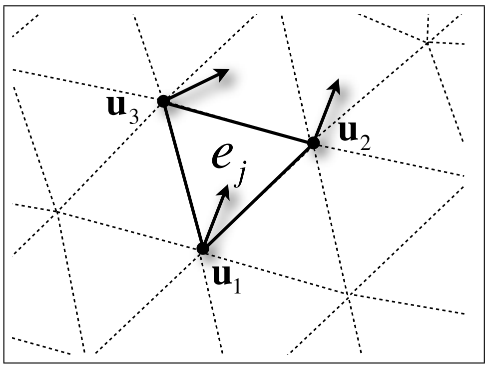

Figure 4: Vectors at the nodes of an element.

$$\boldsymbol{u}(x, y) = \bar{\alpha}x + \bar{\beta}y + \bar{\gamma},$$

where

$$\bar{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{K-1} \end{pmatrix} \quad \bar{\beta} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{K-1} \end{pmatrix} \quad \bar{\gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_{K-1} \end{pmatrix}. \tag{31}$$

The finite element implementation of our functionals approximates the infinite dimensional minimization problem by a finite dimensional (although the number of variables is large) minimization:

$$\mathcal{F}_n(\boldsymbol{u}) \approx \sum_{j=1}^{M} \int_{e_j} \left(\frac{\left|\boldsymbol{u}^j - 2\boldsymbol{u}^j_{n-1} + \boldsymbol{u}^j_{n-2}\right|^2}{2h^2} + \frac{\left|\boldsymbol{u}^j - \boldsymbol{u}^j_{n-1}\right|^2}{2h} + \frac{\left|\nabla \boldsymbol{u}^j\right|^2}{2} \right) dx \tag{32}$$

$$= \sum_{j=1}^{M} \int_{e_j} \sum_{i=1}^{K-1} \left(\frac{\left|u^j_i - 2u^j_{i,n-1} + u^j_{i,n-2}\right|^2}{2h^2} + \frac{\left|u^j_i - u^j_{i,n-1}\right|^2}{2h} + \frac{\left|\nabla u^j_i\right|^2}{2} \right) dx. \tag{33}$$